THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
Tutorial 1

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1. Let $f : [a, b] \to \mathbb{R}$ be a function and $c \in (a, b)$.

   (a) State the definition of differentiability of f at c.

   (b) Recall we have the theorem:

   Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a function and $c \in (a, b)$. Then f is differentiable at c if and only if there exists a function $g : [a, b] \to \mathbb{R}$ which is continuous at c such that*

   $$f(x) - f(c) = g(x)(x - c),$$

   where $g(c) = f'(c)$.

   Using the theorem above, show the following theorem:

   Theorem 2. *(Chain Rule) Let $f : [a, b] \to [d, e]$, $g : [d, e] \to \mathbb{R}$, $c \in (a, b)$, $f(x) \in (d, e)$. Suppose f is differentiable at c and g is differentiable at $f(c)$. Show that the composite function $g \circ f$ is differentiable at c, and that*

   $$(g \circ f)'(c) = g'(f(c)) \cdot f'(c).$$

   (c) What is the problem with the following "proof"?

   $$\begin{aligned}
   &\lim_{x\to c} \frac{g(f(x)) - g(f(c))}{x - c} \\
   &= \lim_{x\to c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c} \\
   &= \lim_{x\to c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \\
   &= \lim_{y\to f(c)} \frac{g(y) - g(f(c))}{y - f(c)} \cdot \lim_{x\to c} \frac{f(x) - f(c)}{x - c}, \text{since } f \text{ is continuous at } c. \\
   &= g'(f(c)) \cdot f'(c)
   \end{aligned}$$

2. (a) State the Mean value theorem.

   (b) Show that if $f : (a, b) \to \mathbb{R}$ has strictly positive derivative at $c \in (a, b)$, then there exists $\delta > 0$ such that for any $c - \delta < x < c < y < c + \delta$, we have $f(x) < f(c) < f(y)$.

   (c) With the same condition, could we get a stronger result that there exists $\delta > 0$ such that for any $c - \delta < x < y < c + \delta$, we have $f(x) < f(y)$? (Hint: Consider the function)

   $$f(x) := \begin{cases} x + 2x^2 \sin(\frac{1}{x}), x \neq 0 \\ 0, x = 0 \end{cases}$$

(d) Let $D \subseteq \mathbb{R}$ be a domain. Using mean value theorem, prove that any function $f : D \to \mathbb{R}$ which is differentiable on D with bounded derivative is uniformly continuous. Hence show that if $f : [a, b]$ has a continuous derivative on $[a, b]$, then f is uniformly continuous and bounded. (One should define left and right derivatives here)

3. We consider some explicit examples.

   (a) Consider $f(x) := \sqrt{x}$ defined on $[0, \infty)$. Show that f is differentiable on $(0, \infty)$ but not at 0.

   (b) Consider $f(x) := x \sin(\frac{1}{x})$ defined on $x \neq 0$ and $f(0) := 0$. Show that f is not differentiable at 0.

   (c) Consider $f(x) := x^2 \sin(\frac{1}{x})$ defined on $x \neq 0$ and $f(0) := 0$. Show that f is differentiable everywhere on $\mathbb{R}$ but $f'(x)$ is not continuous.

   (d) Consider $f(x) := \frac{1}{1+x^2}$ defined on $\mathbb{R}$. Compute its derivative $f'(x)$, and hence show that f is uniformly continuous.

4. **Solution:**

   The function below satisfies $f'(0) = 1 > 0$ but f is not increasing on any neighbourhood of 0.

$$f(x) := \begin{cases} x + 2x^2 \sin(\frac{1}{x}), x \neq 0 \\ 0, x = 0 \end{cases}$$

Proof. Let $\delta > 0$ be arbitrary. Consider two points $x := \frac{1}{2n\pi + \frac{\pi}{2}}$, $y := \frac{1}{2n\pi - \frac{\pi}{2}}$ where n is large enough so that $0 < x < y < \delta$.

Then we compute $f(x) - f(y) = x + 2x^2 - y + 2y^2$. Using $\frac{1}{x} - \frac{1}{y} = \pi$, we can show that $f(x) - f(y) > 0$. Hence f is not increasing on any neighbourhood of 0. □